

Friedmann–Robertson–Walker models do not require zero active mass

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ABSTRACT

The $R_h = ct$ cosmological model has received considerable attention in recent years owing to claims that it is favoured over the standard Λ cold dark matter (Λ CDM) model by most observational data. A key feature of the $R_h = ct$ model is that the zero active mass condition $\rho + 3p = 0$ holds at all epochs. Most recently, Melia (2016b) has claimed that this condition is a requirement of the symmetries of the Friedmann–Robertson–Walker (FRW) spacetime. We demonstrate that this claim is false and results from a flaw in the logic of Melia’s argument.

Key words: cosmology: theory – gravitation

1 INTRODUCTION

The Λ CDM model serves as the basis for the current standard model of cosmology, which provides a good fit to a wide range of cosmological observations. As pointed out by Melia (2003), however, for the best-fit Λ CDM model, the present-day Hubble distance is broadly consistent with ct_0 to within observational uncertainties, where t_0 is the current cosmic epoch. In other words, observations suggest that the universe has expanded by an amount similar to what would have occurred had the expansion rate been constant or, equivalently, that the average acceleration of the universe up to the present epoch is consistent with zero; this is despite the fact that the combination of time-dependent radiation, matter and dark-energy densities $\rho_r(t)$, $\rho_m(t)$ and $\rho_{de}(t)$ should have produced periods of deceleration and acceleration. Another way to describe this finding (Melia 2009) is that, averaged over a Hubble time, the quantity p/ρ , where $\rho = \rho_r + \rho_m + \rho_{de}$ and $p = p_r + p_m + p_{de}$, yields $\langle p/\rho \rangle = -1/3$ to within the observational uncertainties.

In the Λ CDM model, this correspondence is a peculiar coincidence, made all the more striking by the fact that, for the best-fit model, this situation should occur only once in the history of the universe. The fact that we observe this correspondence at the present epoch is therefore most intriguing.

Consequently, it was proposed by Melia (2007, 2009); Melia & Shevchuk (2012) that this correspondence is not co-

incidental, but should be satisfied at all cosmic times t . The physical argument originally presented for this viewpoint was based on applying Birkhoff’s theorem and its corollary to a spherical subregion of a homogeneous and isotropic matter distribution, from which it was claimed that one could identify a gravitational radius $R_h = 2GM/c^2$, given in terms of the Misner–Sharp mass $M = (4\pi/3)R_h^3(\rho/c^2)$ (Misner & Sharp 1964). Moreover, it is easily shown that R_h coincides with the Hubble radius in a spatially-flat universe containing any single fluid component. In particular, it was claimed that Weyl’s postulate requires R_h , and hence the Hubble radius, to be a ‘proper’ distance, i.e. one that is comoving with the cosmological fluid. Imposing this condition on the usual cosmological field equations for an FRW spacetime picks out a unique solution, for which $R_h(t) = ct$ at all cosmic times. This is equivalent to vanishing total active mass, $\rho + 3p = 0$, at all epochs. The resulting cosmological model, known as the ‘ $R_h = ct$ ’ model, has received considerable attention over the last few years, since it has been claimed to be favoured over the standard Λ CDM (and its variant w CDM with $w \neq -1$) by most observational data (Melia & Maier 2013; Wei et al. 2013, 2014a,b, 2015; Melia et al. 2015).

Recent observational data have, however, led to serious criticisms of the $R_h = ct$ model. For example, the model requires the deceleration parameter $q(z) = 0$ at all times, but current data from supernovae and baryon acoustic oscillations strongly disagrees with $q_0 = 0$ at high significance (Bilicki & Seikel 2012), and robust model comparison methods strongly disfavour the $R_h = ct$ model (Shafer 2015). In addition, recent cosmic microwave background (CMB) data from the *Planck* satellite rule out the equivalence of

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the age of the universe to $1/H_0$ at greater than 99 per cent confidence, favouring $R_h = (1.05 \pm 0.02) ct$ at the current epoch (van Oirschot et al. 2015), which undermines a major motivation for the $R_h = ct$ model; note that this result is equivalent to $q_0 = 0.05 \pm 0.02$ (van Oirschot et al. 2015).

In addition to objections based on observations, the validity of the theoretical argument underlying the $R_h = ct$ model has also been criticised by a number of authors (van Oirschot et al. 2010; Lewis & van Oirschot 2012; Mitra 2014), and in particular the validity of the effective equation-of-state parameter $w = -1/3$ (Lewis 2013). These and other criticisms are claimed to have been addressed by Bikwa et al. (2012) and Melia (2012) (see also Melia (2015) and references therein), but the original physical arguments for the model given in Melia (2007, 2009); Melia & Shevchuk (2012) are sufficiently imponderable that it is difficult to draw definite conclusions.

In a recent paper (Melia 2016b), however, Melia presents a much more explicit argument for the zero active mass condition $\rho + 3p = 0$, which he claims is a requirement of the symmetries of the FRW spacetime. In particular, it is claimed that assuming the general, spherically symmetric (but radially varying) metric, solving the Einstein field equations, and then imposing homogeneity and isotropy yields an extra condition, namely vanishing active mass, which is lost if one adopts the usual procedure of first imposing the conditions of homogeneity and isotropy on the metric and then solving the Einstein equations. The purpose of our Letter is to demonstrate that this claim is false, owing to a flaw in the logic of Melia's argument, and hence that FRW models are in fact consistent with non-zero active mass.

2 FRW METRIC AND ZERO ACTIVE MASS

Melia starts with the general spherically symmetric metric in a comoving coordinate system, which we denote by

$$ds^2 = A^2 dt^2 - B^2 dr^2 - R^2 d\Omega^2, \quad (1)$$

where A , B and R are in general functions of both r and t , and first considers the general case, where homogeneity is *not* assumed. Using the Einstein equations, assuming zero cosmological constant, one may derive the Euler and continuity equations

$$\frac{\partial p}{\partial r} = -\frac{1}{A} \frac{\partial A}{\partial r} (\rho + p), \quad (2)$$

$$\dot{\rho} = -(\rho + p) \left(2 \frac{\dot{R}}{R} + \frac{\dot{B}}{B} \right), \quad (3)$$

where a dot denotes differentiation with respect to t . Incidentally, in his equation (13), Melia gives an incorrect form of the continuity equation, $\dot{\rho} = -3(\rho + p)(\dot{R}/R)$, which is valid only in the homogeneous case, but this error has no bearing on the rest of his argument.

Melia then imposes homogeneity and finds from equation (2) that A is independent of r , such that $A = A(t)$. Moreover, as usual, one may also write $B(r, t) = a(t)/\sqrt{1 - kr^2}$ and $R(r, t) = a(t)r$, where $a(t)$ is the scale factor and k is the spatial curvature constant. The Einstein equations then yield the corresponding Friedmann equation

and acceleration equation,

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} \rho A^2 - \frac{k}{a^2} A^2, \quad (4)$$

$$\frac{\ddot{a}}{a} - \frac{\dot{a}}{a} \frac{d \ln A}{dt} = -\frac{4\pi}{3} A^2 (\rho + 3p), \quad (5)$$

which Melia combines into the single equation

$$\frac{d}{dt} \left[\ln \left(\frac{\dot{a}^2}{A^2} \right) \right] = - \left(\frac{k}{a\dot{a}} A^2 + \frac{\dot{a}}{a} \right) \left(1 + \frac{3p}{\rho} \right). \quad (6)$$

Melia then writes $A(t)$ in the following form

$$A^2(t) = h \dot{a}^2 e^{I(t)}, \quad (7)$$

where h is a constant and $I(t)$ is a function defined by the above equation. Substituting (7) into the LHS of (6) and using (4), then gives

$$\frac{dI(t)}{dt} = \frac{8\pi}{3} \frac{a}{\dot{a}} A^2 (\rho + 3p). \quad (8)$$

The flaw in his logic then lies in the following. He asserts that, in order for A to be a constant, as it is in the FRW metric, equation (7) requires *both* \dot{a}^2 and $e^{I(t)}$ to be constant in time. This *incorrect* assertion then leads one to conclude that $dI(t)/dt = 0$ at all times, and that by equation (8), $\rho + 3p = 0$ at all times. He therefore concludes that the FRW metric (for which $A = 1$) requires the zero active mass condition to be satisfied. This assumption is clearly wrong, however, as the RHS of equation (7) can be constant without \dot{a}^2 and $e^{I(t)}$ both being constant.

That equations (7) and (8) can be satisfied for $A = \text{constant}$ and $\rho + 3p \neq 0$ is easily illustrated by a simple example. Let us consider the conventional FRW metric, for which $A = 1$, and specifically the EdS model (which Melia himself uses as an example to support his theory), for which $a(t) \propto t^{2/3}$, $\rho(t) = 1/(6\pi t^2)$ and $p(t) = 0$. We can use these expressions to evaluate the RHS of equation (8) and integrate to find that $e^{I(t)} \propto t^{2/3}$. Since $\dot{a} \propto t^{-1/3}$, the powers of t cancel out on the RHS of (7), showing that A is a constant, as required. It is worth noting that in the above analysis, we have *not* simply imposed $A = 1$ *a priori*, as in the usual procedure for deriving the cosmological field equations, but instead demonstrated that equations (7) and (8), derived by solving the Einstein equations for the general spherically-symmetric metric (1), admit solutions for which A is constant and $\rho + 3p \neq 0$. This counter-example alone thus disproves Melia's central claim.

It is worth making a few further points regarding his argument for zero active mass before moving on to the next part of his argument. First, the expression (8) that Melia presents is strange in that it contains A , which one may eliminate in favour of $I(t)$ using (7). In fact, one can derive the following two expressions for $I(t)$,

$$e^{I(t)} = \frac{3}{h(8\pi \rho a^2 - 3k)}, \quad (9)$$

$$e^{-I(t)} = -\frac{8\pi}{3} h \int_0^t a \dot{a} (\rho + 3p) dt', \quad (10)$$

which make no explicit reference to A . Given forms for ρ and a as functions of t , we can use either of these equations to compute $I(t)$, and then by using equation (7) can find the $A(t)$ implied. We adopt this route in the example studied in the next section. Alternatively, one could start from

a fixed form of A , and work forwards from there. For example, if $A = 1$, then (4) and (5) reduce to the conventional cosmological field equations, and for any solution of them (i.e. for any standard cosmological model) either of the expressions (9) or (10) provides an explicit expression for $I(t)$, which when substituted into (7) yields unity on the LHS. Alternatively, if A is not equal to unity, then the solution for a of (6) will differ from that obtained from the usual cosmological field equations, for which $A = 1$, but this would result in a different expression for $I(t)$, sufficient to combine with \dot{a}^2 in (7) to recover the corresponding expression for A on the LHS.

3 COMOVING AND FREE-FALL FRAMES

Having shown above that having $A = \text{constant}$ in (1) does not require zero active mass, we now address the second part of Melia's argument, in which he claims to provide a justification for requiring A to equal unity; this claim is also incorrect.

In the coordinates defined by (1), he first shows that the 4-velocity of an observer comoving with the fluid is

$$u^0 = 1/A, \quad u^i = 0 \quad (i = 1, 2, 3), \quad (11)$$

where the condition $u^i = 0$ shows that r , θ and ϕ are comoving coordinates. He then points out correctly that a free-falling observer is comoving with the fluid, but goes on to suggest incorrectly that this implies that the proper time of a comoving observer must equal the coordinate time t and hence that $A = 1$. He further notes that if A were a function of t (which, according to his incorrect reasoning addressed above, would be necessary if $\rho + 3p \neq 0$), one might attempt to perform a gauge transformation of the form

$$d\tilde{t} = A dt, \quad (12)$$

which would reduce the metric back to the FRW form, with $g_{\tilde{t}\tilde{t}} = 1$, but he claims that this is not permitted because of the uniqueness of the comoving, free-falling frame.

These claims are easily demonstrated to be false. As we show below, the coordinate time t is allowed to be any function of the proper time of a comoving observer, τ . Therefore, a gauge transformation of the form given by (12) is allowed, and hence it is possible to have A to be dependent on t without any problems.

To illustrate this explicitly, let us consider a cosmology for which the evolution of the scale factor as a function of coordinate time t is that in Melia's own model, namely $a(t) = bt$, where b is a constant. Moreover, again following Melia, we will assume that $k = 0 = \Lambda$, but instead of his assumption concerning $p = -\frac{1}{3}\rho$, we take the cosmic fluid to have zero pressure, so that the underlying *physical* cosmology is the Einstein-de-Sitter (EdS) model.

Substituting $a(t) = bt$ into the continuity equation (3) for the case of a homogeneous universe, one finds

$$\rho = \frac{C}{t^3}, \quad (13)$$

where C is a constant. Substituting this form for the density into our equation for $e^{I(t)}$ in (9), we find that for $k = 0$,

$$e^{I(t)} = \frac{3t}{8\pi h C b^2}, \quad (14)$$

and using equation (7), we find that

$$A(t) = \sqrt{\frac{3t}{8\pi C}}, \quad (15)$$

which is clearly not constant.

It is straightforward to find the rest of the metric components, which are

$$B(r, t) = f'(r)bt, \quad (16)$$

$$R(r, t) = f(r)a(t) = f(r)bt, \quad (17)$$

where $f'(r) = df/dr$ and $f(r)$ is some function of r . With these expressions for A , B and R , we can use the Einstein equations to determine the corresponding stress-energy tensor of the cosmic fluid. As expected, it yields a fluid of density $\rho = C/t^3$, zero pressure, and 4-velocity given by

$$u^0 = 1/A = \sqrt{\frac{8\pi C}{3t}}, \quad u^i = 0 \quad (i = 1, 2, 3). \quad (18)$$

This shows that we are in a frame comoving with the fluid, but there is no requirement that the proper time of a comoving observer must coincide with the coordinate time t .

Indeed, we can find the explicit relationship between t and τ from the first geodesic equation in (18). Solving

$$\frac{dt}{d\tau} = \sqrt{\frac{8\pi C}{3t}}, \quad (19)$$

subject to the boundary condition $t = \tau = 0$ at the big bang, one finds

$$t = (6\pi C \tau^2)^{1/3}. \quad (20)$$

We may verify that this relationship is correct by noting that it leads to the appropriate expression for the density as a function of the proper time of comoving observers in an EdS universe, namely

$$\rho = \frac{1}{6\pi \tau^2}. \quad (21)$$

Hence, the coordinate t here is simply proportional to $\tau^{2/3}$. Note that the specific relation between t and $\tau^{2/3}$ was determined by our choice in $a(t)$; any other choice for $a(t)$ would also yield constant spatial components in the 4-velocity, and the frame would be declared 'comoving', but the t would not (in general) be the proper time of the comoving observer in that frame, and this occurs without any inconsistencies or restrictions.

This one counter-example is sufficient to prove that a gauge transformation of the form (12) is allowed, and A does not necessarily have to be constant. When $A = 1$, we are in the frame of the comoving/freely-falling observer, with the coordinate t equal to their proper time. When $A = A(t)$, the spatial coordinates still are those of the comoving/freely-falling observer, but the coordinate t is simply some function of their proper time, and this function is determined by the specific form of $A(t)$. In this case, one can simply use the gauge transformation given by (12) to bring us back to the conventional FRW metric with $A = 1$, in which the time coordinate is equal to the proper time of the comoving observer.

Finally, we address a related part of Melia's argument (Melia 2016a), in which he claims that it is inconsistent with basic relativity theory to have $A = 1$ in a cosmological model with $\ddot{a} \neq 0$. He bases this claim on the fact that one can

always distinguish between accelerated and inertial frames. In particular, Melia suggests that the accelerated universal expansion should produce a time dilation that is measurable relative to the passage of proper time in the (inertial) free-fall frame, and hence A cannot be unity. The flaw in Melia's argument is that the condition $\ddot{a} \neq 0$ represents a coordinate acceleration rather than a proper acceleration. Any observer comoving with the cosmological fluid follows a geodesic and is hence freely-falling and so does not experience any proper acceleration, and this is perfectly consistent with having $\ddot{a} \neq 0$.

4 CONCLUSIONS

To summarise, Melia's first claim is that for A in (1) to be a constant, one requires $\rho + 3p = 0$. We have shown that this is false, and results simply from a false step in logic, and we have provided an example using the EdS cosmology that demonstrates this. Secondly, Melia claims that A (and hence Φ using Melia's own notation) needs to be constant, by arguing that the free-fall and comoving frames must coincide. We have shown that the two frames can coincide perfectly well even with A not constant; in this case, the coordinate time t is no longer the proper time of comoving observers, but a function of it arising via a simple gauge transformation.

Thus, contrary to the claims presented in Melia (2016b), there is no extra information to be extracted from starting by substituting the general spherically-symmetric metric into the Einstein equations and then imposing homogeneity and isotropy, as compared to the usual route of first imposing homogeneity and isotropy on the metric and then employing the Einstein equations. Hence, the FRW spacetime is perfectly compatible with having $\rho + 3p \neq 0$.

In closing, it is also worth pointing out here that the many claims that the $R_h = ct$ model is favoured over Λ CDM by observational data should also be treated with caution. As discussed in the Introduction, more recent observational data cast doubt on the model's central assumptions, but there exists a further issue related to how the $R_h = ct$ and Λ CDM models have previously been compared. In the Λ CDM model, there is no requirement that $\rho + 3p = 0$. This condition is, however, broadly consistent with much of the observational data, as has been known for some time. Thus, if one merely imposes this additional condition *post-hoc* on the Λ CDM model, to arrive at the $R_h = ct$ model, then any model selection approach will naturally favour the latter. Such analyses have content only if one has a physical reason *a priori* to impose the zero active mass condition. As we have shown, the argument presented in Melia (2016b) for imposing this condition is not valid.

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